INTRODUCTORY BUMPONOMICS: THE TOPOLOGY OF DEFORMATION SPACES OF HYPERBOLIC 3-MANIFOLDS

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Abstract. We survey work on the topology of the space $AH(M)$ of all (marked) hyperbolic 3-manifolds homotopy equivalent to a fixed compact 3-manifold M with boundary. The interior of $AH(M)$ is quite well-understood, but the topology of the entire space can be quite complicated. However, the topology is well-behaved at many points in the boundary of $AH(M)$.

1. Introduction

In this paper we survey recent work on the topology of the space $AH(M)$ of all hyperbolic 3-manifolds homotopy equivalent to a fixed compact 3-manifold with boundary M. The recent resolution of Thurston's Ending Lamination Conjecture (in Minsky [57] and Brock-Canary-Minsky [14, 15]) in combination with the resolution of Marden's Tameness Conjecture (in Agol [1] and Calegari-Gabai [22]), gives a complete classification of the manifolds in $AH(M)$. However, the invariants in this classification vary discontinuously over the space, and we are very far from having a parameterization of $AH(M)$.

The interior $\mathrm{int}(AH(M))$ has been well-understood since the 1970's, due to work of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan, Thurston and others. The components of $\mathrm{int}(AH(M))$ are enumerated by topological data, and each component is a manifold parameterized by natural analytic data. The recent resolution of the Bers-Sullivan-Thurston Density Conjecture assures us that $AH(M)$ is the closure of $\mathrm{int}(AH(M))$.

Since the mid-1990's, there has been a string of results and examples demonstrating that the topology of $AH(M)$ is less well-behaved than originally expected. Anderson and Canary [5] first showed that components of $\mathrm{int}(AH(M))$ can bump, i.e. have intersecting closure, while Anderson, Canary and McCullough [6] characterized exactly which components of $\mathrm{int}(AH(M))$ can bump when M has incompressible boundary. McMullen [55] showed that if $M = S \times I$ (where S is a

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closed surface), then the only component of $\mathrm{int}(AH(M))$ self-bumps, i.e. there is a point in the boundary such that any sufficiently small neighborhood of the point disconnects the interior. Bromberg and Holt [20] showed that self-bumping occurs whenever M contains a primitive essential annulus. Most recently, Bromberg [19] showed that the space of punctured torus groups is not even locally connected. We will survey these results and describe the construction which has been responsible for all the pathological behavior discovered so far.

On the other hand, the topology of the deformation space appears to be well-behaved at most points on its boundary and we will describe some recent results establishing this in a variety of settings.

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2. Definitions

In this section, we will set up some of the notation and introduce some of the definitions used throughout the remainder of the paper.

Let M be a compact 3-manifold whose interior admits a complete hyperbolic structure. We will assume throughout this paper that all surfaces and 3-manifolds are oriented and have non-abelian fundamental group and that all manifolds are allowed to have boundary. Then $AH(M)$ is the space of (marked) hyperbolic 3-manifolds homotopy equivalent to M. More formally, we define

$$AH(M) = \{\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C}) |\ \ \rho \text{ discrete and faithful}\}/\mathrm{PSL}_2(\mathbf{C}).$$

We topologize $AH(M)$ as a subset of the the character variety

$$X(M) = Hom_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))//\mathrm{PSL}_2(\mathbf{C})$$

where $Hom_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$ denotes the space of representations $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ with the property that if g is an element of a rank two abelian subgroup of $\pi_1(M)$, then $\rho(g)$ is either parabolic or the identity. (Here, we are taking the Mumford quotient to guarantee that the quotient has the structure of an algebraic variety, see Kapovich [41] for details.)

If $\rho \in AH(M)$, then $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ is a hyperbolic 3-manifold homotopy equivalent to M. (The elements of $AH(M)$ are really equivalence classes of representations, but as the manifolds associated to conjugate representations are isometric we will consistently blur this distinction.) There is also a homotopy equivalence $h_\rho : M \to N_\rho$ such that

$$(h_\rho)_* : \pi_1(M) \to \pi_1(N_\rho) = \rho(\pi_1(M))$$

agrees with ρ. The homotopy equivalence h_ρ is the "marking" of N_ρ.

Alternatively, we could have defined $AH(M)$ as the space of pairs (N, h) where N is an oriented hyperbolic 3-manifold and $h : M \to N$ is a homotopy equivalence, where we consider two pairs (N_1, h_1) and (N_2, h_2) to be equivalent if there is an orientation-preserving isometry $j : N_1 \to N_2$ such that $j \circ h_1$ is homotopic to h_2. This alternate definition is reminiscent of the classical definition of Teichmüller space as a space of marked Riemann surfaces of a fixed genus.

If $\rho \in AH(M)$, then the *domain of discontinuity* $\Omega(\rho)$ is the largest open subset of $\hat{\mathbf{C}}$ on which $\rho(\pi_1(M))$ acts properly discontinuously. The *limit set* $\Lambda(\rho) = \hat{\mathbf{C}} - \Omega(\rho)$ is its complement. The *conformal boundary* is defined to be $\partial_c N_\rho = \Omega(\rho)/\rho(\pi_1(M))$ and we let

$$\hat{N}_\rho = N_\rho \cup \partial_c N_\rho = (\mathbb{H}^3 \cup \Omega(\rho))/\rho(\pi_1(M)).$$

We say that N_ρ is *convex cocompact*, if $\hat{N}_\rho$ is compact. We say that N_ρ is *geometrically finite* if $\hat{N}_\rho$ is homeomorphic to $M' - P'$ where M' is a compact 3-manifold and P' is a collection of annuli and tori in $\partial M'$. (These definitions are equivalent to more classical definitions, see Marden [51] and Bowditch [11].)

3. The interior of $AH(M)$

In this section, we survey the classical deformation theory of hyperbolic 3-manifolds which gives a beautiful description of the interior $\mathrm{int}(AH(M))$ of $AH(M)$. If the boundary of M consists entirely of tori, then Mostow-Prasad Rigidity [61, 71] implies that any homotopy equivalence between hyperbolic 3-manifolds homotopy equivalent to M is homotopic to an isometry. Therefore, $AH(M)$ has either 0 or 2 points (one gets one point for each orientation of M.) So, we will always assume that the boundary of M has a non-toroidal component.

Marden [51] and Sullivan [74] proved that $\mathrm{int}(AH(M))$ consists exactly of the geometrically finite hyperbolic manifolds $\rho \in AH(M)$ such that every parabolic element of $\rho(\pi_1(M))$ is contained in a free abelian subgroup of rank two. (This is equivalent to requiring that $\hat{N}_\rho$ be homeomorphic to a compact 3-manifold with its toroidal boundary

components removed.) The now classical quasiconformal deformation theory of Kleinian groups shows that geometrically finite hyperbolic 3-manifolds are determined by their (marked) homeomorphism type and the conformal structure on the conformal boundary and that every possible conformal structure arises. For an analytically-oriented discussion of quasiconformal deformation theory, see Bers [8]. For a more topological viewpoint on this material, see chapter 7 of Canary-McCullough [24].

In order to formally state the parameterization theorem for $\mathrm{int}(AH(M))$ we need to introduce some more notation.

We define $\mathcal{A}(M)$ to be the set of oriented, compact, irreducible, atoroidal (marked) 3-manifolds homotopy equivalent to M. More formally, $\mathcal{A}(M)$ is the set of pairs (M', h') where M' is an oriented, compact, irreducible, atoroidal 3-manifold and $h' : M \to M'$ is a homotopy equivalence where two pairs (M_1, h_1) and (M_2, h_2) are considered equivalent if there exists an orientation-preserving homeomorphism $j : M_1 \to M_2$ such that $j \circ h_1$ is homotopic to h_2. We recall that M' is said to be *irreducible* if every embedded 2-sphere in M' bounds a ball and is said to be *atoroidal* if every rank two free abelian subgroup of $\pi_1(M')$ is conjugate to a subgroup of $\pi_1(T')$ for some toroidal boundary component T' of M'.

If $[(M', h')] \in \mathcal{A}(M)$, we define $Mod_0(M')$ to be the group of isotopy classes of homeomorphisms of M' which are homotopic to the identity. We define $\partial_{NT}M'$ to be the non-toroidal components of ∂M and we let $\mathcal{T}(\partial_{NT}M')$ denote the Teichmüller space of all (marked) conformal structures on $\partial_{NT}M'$. Recall that the Teichmüller space of a disconnected surface is simply the product of the Teichmüller spaces of its components, so $\mathcal{T}(\partial_{NT}M')$ is always topologically a cell.

Theorem 3.1. *(Ahlfors, Bers, Kra, Marden, Maskit, Sullivan, Thurston)*

$$\mathrm{int}(AH(M)) \cong \bigcup_{[(M',h')]\in\mathcal{A}(M)} \mathcal{T}(\partial_{NT}M')/Mod_0(M')$$

This identification is quite natural. By the previously mentioned results of Marden and Sullivan, if $\rho \in \mathrm{int}(AH(M))$, there exists a compact, atoroidal, irreducible 3-manifold M_ρ and an orientation-preserving homeomorphism $j_\rho : \hat{N}_\rho \to \mathrm{int}(M_\rho) \cup \partial_{NT}M_\rho$, so we obtain a well-defined marked homeomorphism type $[(M_\rho, j_\rho \circ h_\rho)] \in \mathcal{A}(M)$. If $[(M_\rho, j_\rho \circ h_\rho)] = [(M', h')]$, then we may assume that $M_\rho = M'$ and choose j_ρ so that $j_\rho \circ h_\rho$ is homotopic to h'. With this constraint, j_ρ is well-defined up to post-composition by elements of $Mod_0(M')$ and $\partial_c N_\rho$ is a Riemann surface, so we get a well-defined element of $\mathcal{T}(\partial_{NT}M')/Mod_0(M')$.

Let $\Phi : \mathrm{int}(AH(M)) \to \bigcup_{[(M',h')]\in\mathcal{A}(M)} \mathcal{T}(\partial_{NT}M')/Mod_0(M')$ be the map defined in the previous paragraph. We will discuss the ingredients of the proof that Φ is a homeomorphism. Maskit's Extension Theorem [52] implies that if $\Theta(\rho_1) = \Theta(\rho_2)$, then there exists a quasiconformal map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is conformal on $\Omega(\rho_1)$ and conjugates the action of $\rho_1(\pi_1(M))$ to the action of $\rho_2(\pi_1(M))$. Ahlfors [2] proved that if ρ_1 is geometrically finite, then $\Lambda(\rho_1)$ has measure zero, so we may conclude that φ is conformal. Therefore, Φ is injective. The surjectivity of Φ follows from the Measurable Riemann Mapping Theorem [3] and Thurston's Geometrization Theorem (see Morgan [60]).

If M has incompressible boundary (equivalently if $\pi_1(M)$ is freely indecomposable), then $Mod_0(M')$ is trivial, so each component of $\mathrm{int}(AH(M))$ is topologically a ball. Moreover, Canary and McCullough [24], showed that if M has incompressible boundary, then $\mathcal{A}(M)$ is infinite if and only if M has *double trouble*, i.e. there exist simple closed curves α and β in $\partial_{NT}M$ which are both homotopic to a curve γ in a toroidal boundary component of M but are not homotopic in ∂M. (Alternatively, M has double trouble if and only if there is a thickened torus component of its characteristic submanifold, which intersects the boundary of M in at least two annuli.) We summarize this below.

Corollary 3.2. *If M has incompressible boundary, then* $\mathrm{int}(AH(M))$ *is homeomorphic to a collection of disjoint balls. This collection is infinite if and only if M has double trouble.*

If M has compressible boundary, then typically $\mathcal{A}(M)$ is infinite (see Canary-McCullough [24] for a detailed analysis) and $Mod_0(M)$ is infinitely generated (see McCullough [53].) Maskit [52] showed that $Mod_0(M')$ always acts freely on $\mathcal{T}(\partial_{NT}M')$, so each component of $\mathrm{int}(AH(M))$ is still a manifold.

Examples: 1) If M is homeomorphic to a trivial I-bundle $S \times I$ over a closed orientable surface S of genus $g \geq 2$, then $\mathcal{A}(M)$ has only one element. (Notice that $S \times I$ has the quite rare property that it admits an orientation-reversing self-homeomorphism homotopic to the identity.) The elements of $\mathrm{int}(AH(M))$ are called *quasifuchisan* as they are quasiconformally conjugate to Fuchsian groups. In this case, $\mathrm{int}(AH(M))$ is often denoted $QF(S)$ and $QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(S)$. Concretely, if $\rho \in \mathrm{int}(AH(M))$, then N_ρ is determined by the conformal structure on $\partial_c N_\rho$ and given any conformal structure on $S \times \{0,1\}$ one can construct a complete hyperbolic structure on $S \times (0,1)$ with appropriate conformal boundary.

2) Let S_i denote a surface of genus i with one (open) disk removed. We consider the I-bundle $J_i = S_i \times [0,1]$ and let $\partial_r J_i = \partial S_i \times [0,1]$. Let $\{A_i\}_{i=1}^n$ be a collection of disjoint, parallel, consecutively ordered longitudinal annuli in the boundary of a solid torus V. Then M_n is formed from V and $\{J_1,\ldots,J_n\}$ by attaching $\partial_r J_i$ to A_i. The manifolds M_n are examples of books of I-bundles, see Culler-Shalen [27].

Any irreducible manifold homotopy equivalent to M_n is formed by attaching the $\{J_i\}$ to the $\{A_i\}$ in a different order. To be more precise, if τ lies in the permutation group Σ_n, then one can form M_n^τ from V and $\{J_1,\ldots,J_n\}$ by attaching $\partial_r J_i$ to $A_{\tau(i)}$. One may extend the identity map on $\{J_1,\ldots,J_n\}$ to a homotopy equivalence $h_\tau : M_n \to M_n^\tau$. It is a consequence, see [5], of Johannson's Deformation Theorem [40], that every $(M',h') \in \mathcal{A}(M_n)$ is equivalent to (M_n^τ, h_τ) for some $\tau \in \Sigma_n$.

It is easy to check that if $n \geq 3$, (M_n^τ, h_τ) need not be equivalent to (M_n, id). For example, if $n = 4$ and $\tau = (2\ 3)$ one may check that the boundary components of M_4 have genera 3, 5, 5, and 7, while the boundary components of M_4^τ have genera 4, 5, 5, and 6, so M_4 and M_4^τ are clearly not homeomorphic. On the other hand, if τ is any multiple of the rotation $(1\ 2 \cdots n)$, (M_n^τ, h_τ) is clearly equivalent to (M_n, id). Analyzing the situation more carefully, again using Johannson's Deformation Theorem, one shows that the space $\mathcal{A}(M_n)$ of marked homeomorphism types of manifolds homotopy equivalent to M_n may be identified with cosets of the subgroup generated by $(1\,2 \cdots n)$ in the permutation group Σ_n (see [5]). In particular, $\mathcal{A}(M_n)$ has $(n-1)!$ elements, so $\mathrm{int}(AH(M_n))$ is homeomorphic to a disjoint union of $(n-1)!$ balls. Notice that $M_2 = S_3 \times I$ where S_3 is a closed orientable surface of genus 3.

4. The Density Theorem

Much of the recent work on infinite volume hyperbolic 3-manifolds has culminated in the proof of the Bers-Sullivan-Thurston Density Conjecture which asserts that $AH(M)$ is the closure of its interior. More concretely, the Density Conjecture predicts that every hyperbolic 3-manifold with finitely generated fundamental group is an (algebraic) limit of geometrically finite hyperbolic 3-manifolds.

Density Theorem: *If M is a compact hyperbolizable 3-manifold, then $AH(M)$ is the closure of its interior* $\mathrm{int}(AH(M))$.

There are two approaches to the proof of the Density Theorem. Both approaches make use of the proof of Marden's Tameness Conjecture by Agol [1] and Calegari-Gabai [22] which asserts that every hyperbolic

3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold.

In the classical approach one starts with a hyperbolic manifold $N_\rho \in AH(M)$ and constructs a sequence $\{N_{\rho_i}\}$ in $\mathrm{int}(AH(M))$ such that the end invariants of $\{N_{\rho_i}\}$ converge to those of N_ρ. One then uses convergence results of Thurston [75], Kleineidam-Souto[45], Lecuire [49] and Kim-Lecuire-Ohshika [44] to show that this sequence converges to a limit $N_{\rho'}$. Arguments of Namazi-Souto [63] or Ohshika [66] can then be used to check that $N_{\rho'}$ has the same end invariants as N_ρ. One then invokes the solution of Thurston's Ending Lamination Conjecture [57, 14, 15] to show that $N_\rho = N_{\rho'}$ and thus complete the proof that $N_\rho \in \overline{\mathrm{int}(AH(M))}$.

The other approach makes use of the deformation theory of cone-manifolds developed by Hodgson-Kerckhoff [31, 32] and Bromberg [16]. These ideas were first used by Bromberg [17] to prove Bers' original Density conjecture for hyperbolic manifolds without cusps. This conjecture asserted that any Kleinian surface group with exactly one invariant component of its domain of discontinuity was in the boundary of the "appropriate" Bers slice (see section 8 for the definition of a Bers slice). Brock and Bromberg [12] showed that if M has incompressible boundary and $N_\rho \in AH(M)$ has no cusps, then $N_\rho \in \overline{\mathrm{int}(AH(M))}$. A complete proof of the Density Theorem using these methods was given by Bromberg and Souto [21].

5. Bumping

It would be reasonable to expect that no two components of $\mathrm{int}(AH(M))$ have intersecting closures, since one might expect that any hyperbolic manifold in the closure of a component B would be homeomorphic to a hyperbolic manifold in B. In fact, Jim Anderson and I spent two years attempting to prove this and eventually came up with examples [5] which illustrated that this "bumping" of components can occur. Later, Anderson, Canary and McCullough [6] characterized exactly when two components of $\mathrm{int}(AH(M))$ can bump in the case that M has incompressible boundary.

Formally, we say that two components B and C of $\mathrm{int}(AH(M))$ *bump at* ρ if $\rho \in \overline{B} \cap \overline{C}$. Notice that whenever two components of $\mathrm{int}(AH(M))$ bump, $AH(M)$ is not a manifold.

The first setting in which the phenomenon of bumping was observed, was the books $\{M_n\}$ of I-bundles discussed in section 3.

Theorem 5.1. (Anderson-Canary [5]) *If $n \geq 3$, then any two components of $\mathrm{int}(AH(M_n))$ bump, where $\{M_n\}$ are the books of I-bundles*

constructed in example 2 in section 3. In particular, $AH(M_n)$ *is connected, while* $\text{int}(AH(M_n))$ *has* $(n-1)!$ *components.*

We will outline the construction in the proof of Theorem 5.1 in the section 7. In a finer analysis, Holt [33] showed that there is a single point at which any two components of $\text{int}(AH(M_n))$ bump.

Theorem 5.2. (Holt [33]) *If* $n \geq 3$*, then there exists* $\rho \in AH(M_n)$ *which lies in the boundary of every component of* $\text{int}(AH(M_n))$.

Remark: Holt [33] further observes, that the set of points lying in the closure of every component of $\text{int}(AH(M_n))$ contains an open subset of a (complex) codimension 1 subvariety of $X(M_n)$. It is typical that the "bumping locus" is relatively large.

Anderson, Canary and McCullough [6] later gave a complete characterization of which components of $\text{int}(AH(M))$ can bump in the case when M has incompressible boundary. Roughly, the two components can bump if the two (marked) homeomorphism types differ by cutting along a collection of (primitive) solid tori and rearranging the order in which the complementary pieces are attached. One sees that for M_n any two homeomorphism types differ in this specific way. A solid torus V in M is said to be *primitive* if $V \cap \partial M$ is a non-empty collection of annuli, the inclusion of each annulus of $V \cap \partial M$ into V is a homotopy equivalence and the image of $\pi_1(V)$ in $\pi_1(M)$ is a maximal abelian subgroup.

To illustrate the role of primitivity in Anderson, Canary and McCullough's result, we consider a sequence of manifolds $\{M_n'\}_{n=3}^{\infty}$, again obtained from a solid torus V and I-bundles $\{J_1, \ldots, J_n\}$. This time we let $\{A_1', \ldots, A_n'\}$ be a collection of disjoint, parallel, consecutively ordered annuli in the boundary of a solid torus V such that the inclusion of $\pi_1(A_i')$ into $\pi_1(V)$ is a subgroup of index 3 (i.e. each annulus wrap 3 times around the longitude of V.) We form M_n' by attaching $\partial_r J_i$ to A_i'. It is again the case that any manifold homotopy equivalent to M_n' is obtained by attaching the J_i in a different order and that $\mathcal{A}(M_n')$ has $(n-1)!$ elements. However, the results of [6] imply that no two components of $\text{int}(AH(M_n'))$ bump for any n.

In order to give a precise statement of the results of [6] we must introduce the notion of a primitive shuffle equivalence. In what follows, if M is a compact, irreducible 3-manifold with incompressible boundary, then $\Sigma(M)$ will denote its characteristic submanifold. For complete discussions of the theory of characteristic submanifolds see Jaco-Shalen [39] or Johannson [40]. For a discussion in the setting

of hyperbolizable 3-manifolds, see Canary-McCullough [24] or Morgan [60].

Given two compact irreducible 3-manifolds M_1 and M_2 with nonempty incompressible boundary, a homotopy equivalence $h\colon M_1 \to M_2$ is a *primitive shuffle equivalence* if there exists a finite collection V_1 of primitive solid torus components of $\Sigma(M_1)$ and a finite collection V_2 of solid torus components of $\Sigma(M_2)$, so that $h^{-1}(V_2) = V_1$ and so that h restricts to an orientation-preserving homeomorphism from the closure of $M_1 - V_1$ to the closure of $M_2 - V_2$.

If M is a compact, hyperbolizable 3-manifold with nonempty incompressible boundary, we say that two elements $[(M_1, h_1)]$ and $[(M_2, h_2)]$ of $\mathcal{A}(M)$ are *primitive shuffle equivalent* if there exists a primitive shuffle equivalence $s\colon M_1 \to M_2$ such that $[(M_2, h_2)] = [(M_2, s \circ h_1)]$.

Theorem 5.3. (Anderson-Canary-McCullough [6]) *Let M be a compact, hyperbolizable 3-manifold with nonempty incompressible boundary, and let $[(M_1, h_1)]$ and $[(M_2, h_2)]$ be two elements of $\mathcal{A}(M)$. The associated components of* $\mathrm{int}(AH(M))$ *have intersecting closures if and only if $[(M_2, h_2)]$ is primitive shuffle equivalent to $[(M_1, h_1)]$.*

Combining the work of Anderson, Canary and McCullough [6] with the resolution of the Density Conjecture, one obtains a complete enumeration of the components of $AH(M)$ when M has incompressible boundary. In particular, one completely determines exactly when $AH(M)$ has infinitely many components. Primitive shuffle equivalence gives a finite-to-one equivalence relation on $\mathcal{A}(M)$ and we let $\widehat{\mathcal{A}}(M)$ be the quotient of $\mathcal{A}(M)$ by this equivalence relation.

Corollary 5.4. *If M has incompressible boundary, then the components of $AH(M)$ are in one-to-one correspondence with $\widehat{\mathcal{A}}(M)$. In particular, $AH(M)$ has infinitely many components if and only if M has double trouble.*

Holt [34] refined the analysis of [6] to show that if C_i is a collection of components of $\mathrm{int}(AH(M))$ such that any two components in the collection bump, then then they all bump at a single point.

Theorem 5.5. (Holt [34]) *Let M be a compact, hyperbolizable 3-manifold with nonempty incompressible boundary, and let $\{[(M_i, h_i)]\}_{i=1}^m$ be a collection of elements of $\mathcal{A}(M)$ such that $[(M_i, h_i)]$ is primitive shuffle equivalent to $[(M_1, h_1)]$ for all i. If C_i is the component of* $\mathrm{int}(AH(M))$ *associated to $[(M_i, h_i)]$, then*

$$\bigcap_{i=1}^{m} \overline{C_i} \neq \emptyset.$$

6. Self bumping

McMullen [55] was the first to observe that individual components of $\text{int}(AH(M))$ can self-bump. A component B of $\text{int}(AH(M))$ *self-bumps* at $\rho \in \overline{B}$ if there is a neighborhood V of ρ such that if $\rho \in W \subset V$ is any sub-neighborhood, then $W \cap B$ is disconnected.

Theorem 6.1. (McMullen [55]) *If S is a closed surface, then $QF(S) = \text{int}(AH(S \times I))$ self-bumps.*

Notice that this implies, in particular, that $AH(S \times I)$ is not a manifold. The self-bumping points in [55] are again obtained using the construction from [5] described in section 7.

McMullen's proof made use, in a crucial manner, of the theory of projective structures on surfaces, so did not generalize immediately to manifolds which are not I-bundles. Bromberg and Holt [20] were able to generalize McMullen's result to all manifolds which contain primitive essential annuli. An embedded annulus A in M is said to be *essential* if it is *incompressible*, i.e. $\pi_1(A)$ injects into $\pi_1(M)$, and is not properly homotopic into the boundary ∂M. It is said to be *primitive* if the image of $\pi_1(A)$ in $\pi_1(M)$ is a maximal abelian subgroup of $\pi_1(M)$.

Theorem 6.2. (Bromberg-Holt [20]) *If M contains a primitive essential annulus, then every component of $\text{int}(AH(M))$ self-bumps.*

Notice that there is no assumption that M has incompressible boundary in this theorem. It implies, in particular, that $AH(M)$ is not a manifold if M contains a primitive essential annulus.

Ito has completed an extensive analysis of related phenomena in the space $P(S)$ of complex projective structures on a surface S. There is a natural map $hol : P(S) \to X(S \times I)$ which takes a complex projective structure to its associated holonomy map. Hejhal [30] showed that the map hol is a local homeomorphism and Goldman [29] showed that the components of $Q(S) = hol^{-1}(QF(S))$ are enumerated by the set of weighted multicurves on S. McMullen [55] showed that $QF(S)$ self-bumps by showing that two components of $Q(S)$ can bump. Ito [36, 37] shows that any two components of $Q(S)$ bump, that any component of $Q(S)$ other than the base component self-bumps, and that arbitrarily many components of $Q(S)$ can bump at a single point. (Bromberg and Holt have obtained related results.)

7. The key construction

In this section, we will describe the "wrapping" construction of [5] in the special case of $AH(M_4)$ and show that components of $\mathrm{int}(AH(M_4))$ can bump.

Let $\tau \in \Sigma_4$ be the permutation $(2\ 3)$ and let $\hat{M}_4^\tau$ be the manifold obtained from M_4^τ by removing an open neighborhood of the core curve of V. One may construct an infinite cyclic cover $(\hat{M}_4^\tau)'$ of $\hat{M}_4^\tau$ from an infinite thickened annulus $S^1 \times I \times \mathbb{R}$ by attaching infinitely many copies of each J_i to the outer boundary $S^1 \times \{0\} \times \mathbb{R}$ so that they occur repeatedly in the cyclic order $\ldots J_1, J_3, J_2, J_4, J_1, J_3, \ldots$.... (More concretely, for all $n \in \mathbf{Z}$ and $i = 1, 2, 3, 4$ one attaches a copy of J_i to the thickened annulus by identifying $\partial_r J_i$ with $S^1 \times \{0\} \times [12n + 3\tau(i) - 1, 12n + 3\tau(i) + 1]$ by an orientation-reversing homeomorphism. Vertical translation by 12 on the infinite thickened annulus extends to a homeomorphism of $(\hat{M}_4^\tau)'$ which generates the full group of covering transformations of $(\hat{M}_4^\tau)'$ over M_4^τ.) Let $\pi : (\hat{M}_4^\tau)' \to \hat{M}_4^\tau$ be the covering map.

One then constructs an orientation-preserving embedding $\hat{f}_\tau : M_4 \to (\hat{M}_4^\tau)'$ which takes each J_i homeomorphically to a copy of J_i. (More concretely, one may take J_i to the copy of J_i attached to $S^1 \times I \times [12i + 3\tau(i) - 1, 12i + 3\tau(i) + 1]$.) Let $f_\tau = \pi \circ \hat{f}_\tau$.

Let $\tilde{M}_4^\tau$ be the cover of $\hat{M}_4^\tau$ associated to $(f_\tau)_*(\pi_1(M_4))$. One easily checks that f_τ lifts to an embedding $\tilde{f}_\tau : M_4 \to \tilde{M}_4^\tau$ (since it lifts to an embedding in the intermediate cover $(\hat{M}_4^\tau)'$.) Also, notice that if $i_0 : \hat{M}_4^\tau \to M_4^\tau$ denotes the inclusion map, then $i_0 \circ f_\tau$ is a homotopy equivalence and is homotopic to h_τ (where h_τ is the homotopy equivalence defined in example 2 in section 3).

The central tool in the construction is the generalization of Thurston's Hyperbolic Dehn Filling Theorem [75] to the setting of geometrically finite hyperbolic 3-manifolds.

Let T be a toroidal boundary component of compact 3-manifold M and let (m, l) be a choice of meridian and longitude for T. Given a pair (p, q) of relatively prime integers, we may form a new manifold $M(p, q)$ by attaching a solid torus V to M by an orientation-reversing homeomorphism $g\colon \partial V \to T$ so that, if c is the meridian of V, then $g(c)$ is a (p, q) curve on T with respect to the chosen meridian-longitude system. We say that $M(p, q)$ is obtained from M by *(p, q)-Dehn filling along T*.

If $T = \partial M$ and M is hyperbolizable (i.e. its interior admits a complete hyperbolic structure), then Thurston [75] proved that $M(p, q)$ is

hyperbolizable for all but finitely many choices of (p, q). Bonahon and Otal [10] were the first to observe that you could generalize this result to the setting of geometrically finite hyperbolic 3-manifolds (see also Comar [26], Hodgson-Kerckhoff [31] and Bromberg [18].) Our statement of the resulting Hyperbolic Dehn Filling Theorem is rather convoluted but it essentially says that given a geometrically finite hyperbolic 3-manifold N homeomorphic to $\mathrm{int}(M)$ and (p, q) near to ∞, then there exists a geometrically finite hyperbolic 3-manifold $N(p, q)$ which is "near" to N. The complications in the statement are largely the result of the need to keep careful track of the marking.

Hyperbolic Dehn Filling Theorem: *Let M be a compact, hyperbolizable 3-manifold and let T be toroidal boundary component of M. Let $N = \mathbb{H}^3/\Gamma$ be a geometrically finite hyperbolic 3-manifold admitting an orientation-preserving homeomorphism $\psi : \mathrm{int}(M) \to N$. Further assume that every parabolic element of Γ lies in a rank-two parabolic subgroup. Let $\{(p_n, q_n)\}$ be an infinite sequence of distinct relatively prime pairs of integers.*

Then, for all sufficiently large n, there exists a (non-faithful) representation $\beta_n : \Gamma \to \mathrm{PSL}_2(\mathbf{C})$ with discrete image such that

(1) *$\beta_n(\Gamma)$ is geometrically finite and every parabolic element of $\beta_n(\Gamma)$ lies in a rank-two parabolic subgroup,*
(2) *$\{\beta_n\}$ converges to the identity representation of Γ, and*
(3) *if $i_n : M \to M(p_n, q_n)$ denotes the inclusion map, then for each n, there exists an orientation-preserving homeomorphism $\psi_n : \mathrm{int}(M(p_n, q_n)) \to \mathbb{H}^3/\beta_n(\Gamma)$ such that $\beta_n \circ \psi_*$ is conjugate to $(\psi_n)_* \circ (i_n)_*$.*

In order to apply the Hyperbolic Dehn Filling Theorem in our setting we need to make a couple more topological observations. We choose a meridian-longitude system for the toroidal boundary component T of $\hat{M}_4^\tau$ so that the meridian bounds a disk in M_4^τ and the longitude bounds an essential annulus A in $\hat{M}_4^\tau$. First, one checks that $\hat{M}_4^\tau(1, n)$ is homeomorphic to M_4^τ for all n. (The easiest way to see this is to note that Dehn twisting n times about A takes the $(1, 0)$ curve on T to the $(1, n)$ curve and observe that $\hat{M}_4^\tau(1, 0) = M_4^\tau$.) Then one similarly notices that $(\hat{M}_4^\tau(1, n), i_n \circ f_\tau)$ is equivalent to (M_4^τ, h_τ) in $\mathcal{A}(M_4)$ for all n, where $i_n : \hat{M}_4^\tau \to M_4^\tau(1, n)$ is the inclusion map and h_τ is the homotopy equivalence defined in example 2 in section 3.

Now let $N = \mathbb{H}^3/\Gamma$ be a geometrically finite hyperbolic 3-manifold admitting an orientation-preserving homeomorphism $\psi : \mathrm{int}(\hat{M}_4^\tau) \to N$. (We further require that all parabolic elements of Γ lie in rank

two parabolic subgroups.) In the Hyperbolic Dehn Filling Theorem we choose $(p_n, q_n) = (1, n)$ for all n and let $\beta_n : \Gamma \to \mathrm{PSL}_2(\mathbf{C})$ be the resulting sequence of representations.

For each n, define $\rho_n = \beta_n \circ \psi_* \circ (f_\tau)_*$. The sequence $\{\rho_n\}$ converges to $\rho = \psi \circ (f_\tau)_*$. Since $\beta_n \circ \psi_*$ is conjugate to $(\psi_n)_* \circ (i_n)_*$, we see that ρ_n is conjugate to $(\psi_n)_* \circ (i_n)_* \circ (f_\tau)_*$. Since $i_n \circ f_\tau$ is a homotopy equivalence and ψ_n is a homeomorphism, ρ_n is a discrete faithful representation with image $\pi_1(N_n) = \beta_n(\hat{\Gamma})$. Since $(\hat{M}_4^\tau(1, n), i_n \circ f_\tau)$ is equivalent to (M_4^τ, h_τ), it follows that $(\hat{N}_{\rho_n}, h_{\rho_n})$ is equivalent to (M_4^τ, h_τ) for all n. In particular, ρ_n lies in the component of $\mathrm{int}(AH(M_4))$ associated to (M_4^τ, h_τ) for all n. On the other hand, since f_τ lifts to an embedding, Bonahon'sTameness Theorem [9] and results of McCullough-Miller-Swarup [54] imply that N_ρ is homeomorphic to $\mathrm{int}(M_4)$. So, $\{\rho_n\}$ is an example of a sequence in $AH(M_4)$ where the homeomorphism type changes in the limit.

It remains to show that ρ lies in the closure of the component of $\mathrm{int}(AH(M_4))$ associated to (M_4, id). We may accomplish this by modifying the above construction. We first construct a geometrically finite hyperbolic 3-manifold N' which admits an orientation-preserving homeomorphism $\psi' : \mathrm{int}(\hat{M}_4) \to N'$ (where $\hat{M}_4$ is the manifold obtained from M_4 by removing an open neighborhood of the core curve of V). We further require that there exists an embedding $f : M_4 \to \hat{M}_4$ such that $i'_n \circ f : M_4 \to \mathrm{int}(\hat{M}_4(1, n))$ is a homotopy equivalence which is homotopic to an orientation-preserving homeomorphism for all n (where $i'_n : \hat{M}_4 \to \hat{M}_4(1, n)$ is the inclusion map). Finally, we require that ρ is conjugate to $\psi'_* \circ f_*$. (To construct N' from M_ρ, we may normalize so that there is a maximal abelian subgroup of $\rho(\pi_1(M_4))$ generated by the Möbius transformation $z \to z + 1$. If one considers the group Γ' generated by $\rho(\pi_1(M_4))$ and $z \to z + ri$ for a large enough real value of r, then one may take $N' = \mathbb{H}^3/\Gamma'$.) One then applies the Hyperbolic Dehn Filling Theorem just as before to obtain a sequence $\{\rho'_n\}$ of representations lying in the component of $\mathrm{int}(AH(M_4))$ associated to (M_4, id) and converging to ρ. Alternatively, one may simply apply a theorem of Ohshika [64].

It should be fairly clear that the above construction works for all n and all $\tau \in \Sigma_n$. In fact, it works for all primitive shuffle equivalences, see [6].

This construction is also responsible for the phenomenon of self-bumping. Recall that $M_2 = S_3 \times I$ where S_3 is a closed surface of genus 3. We illustrate the modifications of the "wrapping construction" necessary to produce a self-bumping point in the boundary of $QF(S_3)$.

Let $\hat{M}_2$ be obtained from M_2 by removing an open neighborhood of the core curve of V and let $(\hat{M}_2)'$ be the infinite cyclic cover obtained by gluing alternating copies of J_1 and J_2 to an infinite thickened annulus. We construct an orientation-preserving embedding $\hat{f} : M_2 \to (\hat{M}_2)$ which maps J_1 to the "first copy" of J_1 and maps J_2 to the "second" copy of J_2 (so that there are copies of J_2 and J_1 which lie "between" the image of J_1 and the image of J_2). We then let $f = \pi \circ \hat{f}$, where $\pi : (\hat{M}_2)' \to \hat{M}_2$ is the covering map, and apply Thurston's Hyperbolic Dehn Filling Theorem as before, to produce a sequence $\{\rho_n\}$ in $QF(S_3)$ which converges to a self-bumping point $\rho \in \partial AH(M_2)$. (The assertion that ρ is a self-bumping point is not obvious and requires a clever proof, see McMullen [55] or Bromberg-Holt [20]).

8. Relative deformation spaces and the failure of local connectivity

There are various naturally defined subsets of $AH(M)$ which have played a prominent role in the theory of Kleinian groups. Most simply, one may require that certain elements of $\pi_1(M)$ be mapped to parabolic elements. In this setting, it is natural to introduce the language of pared manifolds. We refer the reader to Morgan [60] and Canary-McCullough [24] for a more extensive discussion of pared manifolds.

Let M be a compact, orientable, irreducible 3-manifold with nonempty boundary which is not a 3-ball, and let $P \subseteq \partial M$. We say that (M, P) is a *pared* 3-manifold if

(1) Every component of P is an incompressible torus or annulus,
(2) every noncyclic abelian subgroup of $\pi_1(M)$ is conjugate into the fundamental group of a component of P, and
(3) every map $\varphi\colon (S^1 \times I, S^1 \times \partial I) \to (M, P)$ which induces an injection on fundamental groups is homotopic, as a map of pairs, to a map ψ such that $\psi(S^1 \times I) \subset P$.

We can then define the relative deformation space $AH(M, P)$ to be the set of (conjugacy classes) of representations $\rho \in AH(M)$ such that if Q is any component of P then $\rho(\pi_1(Q))$ consists of parabolic elements. $AH(M, P)$ lies in a relative character variety $X(M, P)$. We may define the set $\mathcal{A}(M, P)$ to be the set of marked, oriented pared manifolds homotopy equivalent to (M, P) up to orientation preserving pared homeomorphisms. Again the interior of $AH(M, P)$ (within $X(M, P)$) can be identified with

$$\bigcup_{(M',P') \in \mathcal{A}(M,P)} \mathcal{T}(\partial M' - P')/Mod_0(M', P')$$

where $Mod_0(M', P')$ is the group of isotopy classes of pared self-homeomorphism of (M', P') which are pared homotopic to the identity (see Canary-McCullough [24] for more details.)

If S is a compact surface with boundary, then the interior $QF(S)$ of $AH(S \times I, \partial S \times I)$ is again the space of quasifuchsian groups which are quasiconformally conjugate to a cofinite area Fuchsian group uniformizing the interior S^0 of S and is naturally identified with $\mathcal{T}(S^0) \times \mathcal{T}(S^0)$. In the case that S^0 is a once-punctured torus, $AH(S \times I, \partial S \times I)$ is known as the space of punctured torus groups. Bromberg [19] proved that the space of punctured torus groups is not locally connected.

Theorem 8.1. (Bromberg [19]) *The space of punctured torus groups is not locally connected.*

Bromberg's proof suggests that many deformation spaces and relative deformation spaces are not locally connected. Magid [50] recently observed that you can use Bromberg's result to prove that an infinite family of relative deformation spaces is not locally connected. The simplest case of his result is the following.

Theorem 8.2. (Magid [50]) *If S is a closed surface, $M = S \times I$, and P is a collection of non-parallel incompressible annuli in $S \times \{1\}$ such that at least one component of $S \times \{1\} - P$ is homeomorphic to a once-punctured torus, then $AH(M, P)$ is not locally connected.*

More generally, Magid's result shows that $AH(M, P)$ fails to be locally connected whenever there is a separating essential annulus A in M with one boundary component lying in a component P_0 of P such that the closure X of one component of $M - A$ is homeomorphic to $S \times I$ where S^0 is a once-punctured torus (by a homeomorphism taking $\partial S \times I$ to A) and that X intersects P only in P_0.

Results of Holt-Souto [35] and Evans-Holt [28] combine to show that the set of self-bumping points, and hence the set of points where the deformation space is not locally connected, is not dense in the boundary of the space of punctured torus groups. In particular, they show that there exists some constant $\epsilon_0 > 0$ such that if ρ is a punctured torus group and every closed geodesic in $\partial_c N_\rho$ which is homotopic into a cusp in $\widehat{N}_\rho$ has length at least ϵ_0, then the space of punctured torus groups does not self-bump at ρ.

Ito [38] has quite recently given a complete description of the self-bumping points in the boundary of the space of punctured torus groups. In particular, he shows that all self-bumping points arise from the construction described in section 7. Ohshika [67] has generalized many

of Ito's result to the setting of all quasifuchsian spaces. Moreover, he is able to show that there is no self-bumping at many points in the boundary of quasifuchsian space.

Another natural and well-studied class of deformation spaces are the Bers slices which sit inside the quasifuchsian spaces. We recall that if S is a compact surface, then its associated space $QF(S)$ of quasifuchsian groups is naturally identified with $\mathcal{T}(S^0) \times \mathcal{T}(S^0)$. If $\sigma \in \mathcal{T}(S^0)$, then the associated *Bers slice* B_σ is the set $\mathcal{T}(S^0) \times \{\sigma\}$ of quasifuchsian groups, whose bottom conformal boundary component has conformal structure σ. Bers [7] proved that the closure of any Bers slice is compact, so it is natural to study the topology of the closure of such a component. Minsky [56] proved that if S^0 is a once-punctured torus, then the closure of any Bers slice is homeomorphic to a disk.

Theorem 8.3. (Minsky [56]) *If S^0 is a once-punctured torus, then the closure in the space of punctured torus groups of any Bers slice in $QF(S)$ is homeomorphic to a closed disk.*

Remark: Minsky [56] also showed that the closure of the Maskit slice is homeomorphic to a closed disk. Komori [46] has established the same result for the Earle slice and Komori and Parkkonen [47] show that Bers-Maskit slices have the disk as closure.

If the complex dimension of the Bers slice is greater than 1, then very little is known about the topology of the closure of a Bers slice. However, Kerckhoff and Thurston [43] showed that, in this setting, there exist Bers slices B_σ and $B_{\sigma'}$ such that the natural homeomorphism between B_σ and $B_{\sigma'}$ does not extend to a homeomorphism of their closures. They use this to show that there exist Bers slices such that the action of the mapping class group does not extend continuously to the closure. Kerckhoff and Thurston's results were the first indication that the topology of closures of Bers slices must be "complicated."

9. Untouchable points

In this section, we will describe joint work with Jeff Brock, Ken Bromberg and Yair Minsky [13] which shows that the topology of $AH(M)$ is well-behaved at "most" points in the boundary of $AH(M)$ in many cases.

A point $\rho \in \partial AH(M)$ is said to be *untouchable* if there is no bumping or self-bumping at ρ. Notice that $AH(M)$ is locally connected at all untouchable points.

Theorem 9.1. ([13]) *Suppose that M has incompressible boundary. If $\rho \in \partial AH(M)$ and $\rho(\pi_1(M))$ contains no parabolic elements, then ρ is untouchable.*

If ∂M contains no tori, then such points are generic in $\partial AH(M)$.

The proof of the non-bumping portion of this result is a straightforward application of earlier work and we will provide a brief outline of the argument. One first notes that results of Thurston [75], Canary [23] and Anderson-Canary [4] imply that if $\{\rho_i\} \subset \text{int}(AH(M))$ converges to ρ, then $\{N_{\rho_i}\}$ converges geometrically to N_ρ (i.e. larger and larger portions of N_ρ look increasingly like portions of N_{ρ_i}.) Results of Thurston [75], Canary-Minsky [25] and Ohshika [65], then imply that N_ρ is homeomorphic to N_{ρ_i} (by a homeomorphism in the homotopy class determined by $\rho \circ \rho_i^{-1}$) for all large i. Hence the (marked) homeomorphism type is locally constant at ρ, so there is no bumping at ρ.

The proof that there is no self-bumping at ρ is somewhat more involved. One begins by showing that if $\{\rho_i\}$ converges to ρ, then the end invariants of $\{N_{\rho_i}\}$ converge to those of N_ρ. (This is not always the case, so we are also using our restrictions on ρ here.) We then consider any two sequences $\{\rho_i\}$ and $\{\rho_i'\}$ in $\text{int}(AH(M))$ converging to ρ and construct paths $\alpha_i : I \to \text{int}(AH(M))$ joining ρ_i to ρ_i', so that the sequence $\{\alpha_i\}$ converges to the constant path with image ρ. To establish the convergence of these paths, we use the same convergence results and arguments as used in the proof of the Density Theorem as well as the resolution of the Ending Lamination Conjecture.

If we allow N_ρ to have cusps, then $\{N_{\rho_i}\}$ need not converge geometrically to N_ρ in the above argument, so to rule out bumping we must place additional restrictions on ρ. We say that ρ is *quasiconformally rigid*, if every component of $\partial_c N_\rho$ is a thrice-punctured sphere. Notice that this includes the case that $\partial_c N_\rho$ is empty. (We call such representations quasiconformally rigid, since Sullivan's rigidity theorem [73] guarantees that any representation quasiconformally conjugate to ρ is in fact conformally conjugate.)

Theorem 9.2. ([13]) *If $\rho \in \partial AH(M)$ is quasiconformally rigid, then there is no bumping at ρ.*

Remark: There is a related result of Anderson, Canary and McCullough (Corollary 8.2 in [6]) in the setting where M has incompressible boundary which is much stronger.

We provide a brief outline of the argument, which is again largely an exercise in applying previously developed technology. Suppose that

$\{\rho_i\} \subset \mathrm{int}(AH(M))$ converges to ρ and $\{N_{\rho_i}\}$ converges geometrically to N_∞. Then there exists a covering map $\pi : N_\rho \to N_\infty$. Results of Anderson-Canary [4] and Canary [23] imply that there exists a compact core C for N_ρ which embeds in N_∞ (via π). For large i, one can pull-back $\pi(C)$ to obtain a compact core C_i for N_{ρ_i} such that C_i is homeomorphic to C (by a homeomorphism in the homotopy class determined by $\rho_i \circ \rho^{-1}$). The main result of McCullough-Miller-Swarup [54] imply that C_i is homeomorphic to N_{ρ_i} (by a homeomorphism homotopic to inclusion). Therefore, N_{ρ_i} is homeomorphic to N_ρ for all large enough i (again by a homeomorphism in the correct homotopy class). So, there is no bumping at ρ.

To rule out self-bumping, we need to further restrict our setting. We recall that a compact, irreducible manifold is said to be *acylindrical* if it does not contain any essential annuli. (We note that there exist manifolds which are not acylindrical, but do not contain any *primitive* essential annuli.)

Theorem 9.3. ([13]) *If M is acylindrical or homeomorphic to $S \times I$ (where S is a closed surface) and $\rho \in \partial AH(M)$ is quasiconformally rigid, then there is no self-bumping at ρ. So, ρ is untouchable.*

The proof of this theorem is much more involved and makes use in a key way of the techniques of proof of the Ending Lamination Conjecture, in particular Minsky's a priori bounds [57], as well as Thurston's bounded image theorem (see Kent [42]).

Remark: In the case that $M \cong S \times I$, our results overlap substantially with results of Ohshika [67]. We refer the reader to Ohshika's paper for the detailed definitions.

Theorem 9.4. (Ohshika [67]) *Suppose that S is a compact hyperbolic surface and $\rho \in \partial AH(S \times I, \partial S \times I)$.*

(1) *If $\partial_c N_\rho$ has one component homeomorphic to S^0 and all other components are thrice-punctured spheres and every rank one cusp of N_ρ (which is not homotopic to a component of ∂S) abuts a geometrically infinite end, then there is no self-bumping at ρ.*
(2) *If ρ is quasiconformally rigid and every rank one cusp of N_ρ (which is not homotopic to a component of ∂S) abuts a geometrically infinite end, then there is no self-bumping at ρ.*

His techniques are quite different and make deep use of Soma's work [72] on geometric limits of quasifuchsian hyperbolic 3-manifolds.

Ohshika and Soma [68] have recently extended Soma's work on geometric limits.

10. Questions and conjectures

In this final section, we will collect conjectures and questions about the topology of deformation spaces of hyperbolic 3-manifolds. As we still understand very little, much remains to study and we limit ourselves to a few of the more obvious questions.

10.1. **Local connectivity.** Bromberg has conjectured that the failure of local connectivity is a fairly widespread phenomenon. As a first step, one might hope to show that it holds for all spaces of surface groups.

Conjecture 10.1. (Bromberg [19]) *If S is any compact surface, then $AH(S \times I, \partial S \times I)$ is not locally connected.*[1]

Bromberg's proof of Theorem 8.1 makes essential use of the wrapping construction described in section 7. However, he expects that local connectivity should fail even in settings where one cannot perform this construction, e.g. for Bers slices.

Conjecture 10.2. (Bromberg [19]) *If S is any compact hyperbolic surface whose interior S^0 is not a once-punctured torus, thrice-punctured sphere or 4-times punctured sphere, then the closure of any Bers slice in $QF(S)$ is not locally connected.*

As Bromberg points out, these two specific conjectures strongly suggest the following more dramatic conjecture

Conjecture 10.3. *If M is any compact hyperbolizable 3-manifold with a non-toroidal boundary component, then $AH(M)$ is not locally connected.*

It should be pointed out that if M does not contain a primitive essential annulus, then we don't even know that M is not a manifold, so one might begin with the following easier conjecture.

Conjecture 10.4. *If M is any compact hyperbolizable 3-manifold with a non-toroidal boundary component, then $AH(M)$ is not a manifold.*

[1]Conjecture 10.1 has recently been established by Aaron Magid when S is a closed orientable surface of genus at least two.

10.2. **The compressible case.** In the case where M has incompressible boundary, Corollary 5.4 gives a complete enumeration of the components of $AH(M)$, but the situation where M is allowed to have compressible boundary is still quite mysterious. It is easy to use the constuction in section 7 to show that various components of $\mathrm{int}(AH(M))$ bump. Bromberg and Holt's result, Theorem 6.2, already applies to show that if M has compressible boundary, and is not obtained from one or two thickened tori by adding a single 1-handle, then every component of $\mathrm{int}(AH(M))$ self-bumps, since M will contain a primitive essential annulus in these cases.

One would like to give a complete enumeration of the components of $AH(M)$ whenever M has compressible boundary. In the incompressible boundary situation, this is accomplished by showing that the construction in section 7 is responsible for all bumping phenomena.

Problem 10.5. *Give a complete enumeration of the components of $AH(M)$ in terms of topological data.*

One might hope to begin by establishing the following simpler conjecture, which follows from Theorem 5.3 in the case where M has incompressible boundary.

Conjecture 10.6. *$AH(M)$ has finitely many components if and only if its interior has finitely many components.*

A first step in the proof of this conjecture might be to show that it is not possible for infinitely many components of $\mathrm{int}(AH(M))$ to accumulate at a single point.

10.3. **The fractal nature of deformation spaces.** Closures of Bers slices in punctured torus spaces admit natural embeddings in $\mathbf{C}$. The beautiful pictures of these embeddings, see Komori-Sugawa-Wada-Yamashita [48], Yamashita [76], and Mumford-Series-Wright [62], suggest that their boundaries are quite "fractal" in nature. One might guess that the boundaries of these slices have Hausdorff dimension strictly between 1 and 2.

Question 10.7. *What can one say about the Hausdorff dimension of the boundary of a Bers slice of a punctured torus?*

Miyachi [58, 59] has shown that there is a countable dense set of "cusps" in the boundary of a Bers slice of a punctured torus. Parkkonen [69] has further analyzed the shapes of these cusps in the related setting of a Maskit slice. Parkkonen [70] has also studied the shape of Schottky space near certain boundary points. (Schottky space is $\mathrm{int}(AH(H_g))$ where H_g is the handlebody of genus g.)

Of course, one would like to study more general classes of deformation spaces in the future.

10.4. **Components of $AH(M)$ and components of the character variety.** It is known that $AH(M)$ can be disconnected, but it is not known whether or not the different components of $AH(M)$ can lie in different components of the character variety.

Question 10.8. *Is it possible that different components of $AH(M)$ lie in different components of the character variety?*

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